Degenerate states of narrow semiconductor rings in the presence of spin orbit coupling: Role of time-reversal and large gauge transformations

S.-R. Eric Yang*

Physics Department, Korea University, Seoul Korea

The electron Hamiltonian of narrow semiconductor rings with the Rashba and Dresselhaus spin orbit terms is invariant under time-reversal operation followed by a large gauge transformation. We find that all the eigenstates are doubly degenerate when integer or half-integer quantum fluxes thread the quantum ring. The wavefunctions of a degenerate pair are related to each other by the symmetry operation. These results are valid even in the presence of a disorder potential. When the Zeeman term is present only some of these degenerate levels anticross.

I. INTRODUCTION

Energy spectrum of a quantum ring threaded by a magnetic flux is important since several interesting properties of the ring depend on it[1, 2]. The energy spectrum is periodic with the period equal to the quantum unit of flux $\Phi_0 = hc/e$, where the elementary charge $e > 0$. This effect is a consequence of the Aharonov Bohm effect. The angular momentum of an electron in the groundstate of a quantum ring in a magnetic field can be non zero, as can be seen in the energy spectrum of Fig.1(a). This fact is closely related to the fascinating physics of the persistent currents in mesoscopic rings[1, 2, 3]. The change of groundstate angular momentum about the quantization axis from 0 to −1 is a decisive feature distinguishing quantum rings from quantum dots. Such angular momentum transitions were observed in self assembled quantum rings[4]. Effects of spin-orbit terms have been studied in mesoscopic rings[5, 6, 7]. Currently there is renewed interest in them in semiconductor quantum rings. This is because electron spin may be controlled by spin-orbit terms. Such a control would be valuable for spintronics, quantum information, and spin qubits[8]. Recently several effects of spin orbit coupling on optical[9, 10] and transport[11, 14, 15, 16] properties of semiconductor quantum rings have been investigated. A spin filter[12] and a qubit[13] have been also proposed. In this paper we investigate the energy degeneracy of self-assembled semiconductor rings on the magnetic flux in the presence of spin orbit interactions. We show how *large gauge* and *time reversal* transformations can be used to determine the degenerate properties of the single electron energy spectrum of such a system.

Before we give a summary of our main results we comment on some basic properties of large gauge transformations, spin orbit terms in semiconductors, and time reserval operation. Consider an ideal one-dimensional ring without spin orbit coupling and with the radius much larger than the width of the ring. Suppose it is threaded by a magnetic flux Φ (the direction of the magnetic field is chosen to be along the z-axis). It has rotational invariance about the z-axis that goes through the center of the ring. In addition it is invariant under a *large* gauge transformation, which transforms wavefunctions as follows. When the flux is increased from Φ to $\Phi + \Delta\Phi$ the electron wavefunction changes as

$$\Psi' = e^{-i\Delta f\phi}\Psi, \tag{1}$$

where *only* integer values of $\Delta f = \Delta\Phi/\Phi_0$ are allowed[17]. The azimuthal angle of the cylindrical coordinate system is ϕ. Although time reversal symmetry is broken when a flux threads the ring spin invariance is present. Each energy level is thus doubly spin degenerate. However, when the dimensionless flux $f = \Phi/\Phi_0$ is a half-integer or integer each energy level can be 4 times degenerate, as shown in Fig.1(a).

In semiconductors when the external confinement potential breaks the inversion symmetry the Rashba spin orbit term is relevant. When the crystal potential itself breaks the inversion symmetry the Dresselhaus spin orbit term is relevant. In II-VI semiconductors the Rashba term is expected to be larger than the the Dresselhaus coupling. In III-V semiconductors, such as GaAs, the opposite is true[18]. Spin orbit terms break both spin and angular momentum symmetries and the double degeneracy of the ideal one-dimensional ring is broken.

In the absence of a vector potential and the Zeeman term the spin-orbit Hamiltonian is invariant under time reversal symmetry: $\vec{k} \rightarrow -\vec{k}$ and $\vec{S} \rightarrow -\vec{S}$, where $\vec{k}$ and $\vec{S}$ are the wavevector and spin of an electron. The time reversal operator is $K = -i\sigma_y C$, where the operator C stands for complex conjugation and σ_y is one of the Pauli spin matrices $\sigma_{x,y,z}$. If the Hamiltonian is invariant under time reversal operation $\Psi \rightarrow \Psi'$ then the wavefunctions Ψ and Ψ' are degenerate and orthonormal. Note that $K^2\Psi = -\Psi$. This symmetry is the origin of Kramers' double degeneracy in quantum dots[19]

We explore the interplay between large gauge transforation, time reversal operation, and spin orbit coupling in semiconductor rings. We show in the presence of the Rashba and/or Dresselhaus terms that, although spin double degeneracy is lifted, all the energy levels at in-
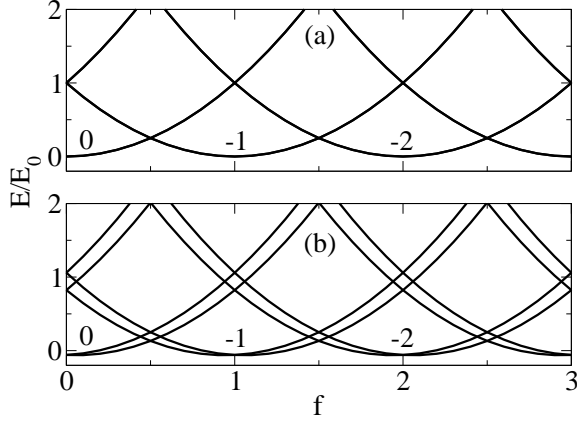
*eyang@venus.korea.ac.kr

FIG. 1: (a) The single electron energy of an ideal one-dimensional ring threaded by a magnetic flux Φ is $E_n(f) = E_0(n+f)^2$, where E_0 is the energy scale $E_0 = \frac{\hbar^2}{2m^*R^2}$ with the electron effective mass m^* and the radius of the ring R. The electron with this energy has the wavefunction $\Psi(\phi) = \frac{1}{\sqrt{2\pi}}e^{in\phi}$, where n is the z-component of the angular momentum. Each energy curve is spin degenerate and parabolic. It takes zero value at integer values $f = -n$. The numbers $0, -1, -2$ near the curves give the z-component of angular momentum of the electron states. (b) Energy levels of a narrow ring of (a) in the presence of the Rashba term which lifts spin degeneracy. The energy scale associated with the Rashba constant is set to $E_R = 0.5E_0$. The numbers near the curves label the energy curves; they are no longer z-components of angular momentum since the rotational symmetry is broken. At integer and half integer values of f the energy levels are doubly degenerate.

teger and half integer values of f are *doubly* degenerate, see Fig.1(b). This is because at these particular values of f a large gauge transformation followed by time reversal operation[19] is a *symmetry operation* of the Hamiltonian. This result is valid even in the presence of a disorder potential and in non-circular rings. The wavefunctions of a degenerate pair are related to each other by the symmetry operation. When only the Rashba term is present analytical solutions of eigenstates are possible. In this case we find, as shown in Fig.1(b), that the energy curve $E_n(f)$ of the ideal ring splits into curves $E_{m,L}(f)$ and $E_{n,U}(f)$, where m is an integer. As a function of f the energy $E_{m,L}(f)$ or $E_{n,U}(f)$ describes a parabola. We find that degeneracy occurs when $E_{m,L}(f) = E_{n,U}(f)$, which leads to

$$n = -m - 2f. \tag{2}$$

Since m and n are both integers f must be half integers or integers. Unlike the previous case of the ideal ring the total degeneracy is now only two. This relation between m and n can be understood in terms of time reversal and gauge transformations, as we demonstrate below. When the Zeeman term is present some degenerate level anti-cross, as shown in Fig.3.

II. IDEAL ONE-DIMENSIONAL RING

Before we analyse the narrow semiconductor rings it would be instructive to review the basic properties of the ideal one-dimensional ring[3] in the absence of spin orbit coupling. The periodic nature of the energy spectrum of an ideal one-dimensional ring as a function of f can be understood by considering invariance under large gauge transformations. Consider an electron moving in an ideal one-dimensional ring with radius R on the x-y plane. The Hamiltonian is $H_0 = \frac{1}{2m^*}(\vec{p} + \frac{e}{c}\vec{A})^2$, where m^* and $\vec{A}$ are the effective mass of the electron and the vector potential. The electron is under the influence of the vector potential due to a solenoid at the center of the ring. The vector potential given by $\vec{A} = A_\phi \hat{\phi}$, where $A_\phi = \frac{\Phi}{2\pi R}$ and Φ is the total magnetic flux through the ring. Without spin orbit terms the Hamiltonian commutes with spin operator and each energy level is *doubly spin degenerate*. The wavefunction $\Psi(\phi)$ satisfies

$$\frac{\hbar^2}{2m^*R^2}(\frac{1}{i}\frac{\partial}{\partial\phi} + f)^2\Psi = E\Psi, \tag{3}$$

The solutions $\Psi(\phi)$ are given by $\frac{1}{\sqrt{2\pi}}e^{in\varphi}$, where n is an integer. A gauge transformation $\vec{A}' = \vec{A} + \nabla\chi(\vec{r})$ leads to the transformed wavefunction $\Psi' = e^{-\frac{e}{\hbar c}\chi(\vec{r})}\Psi$. When an integral multiple of quantum unit of flux is added to the ring the wavefunction satisfies

$$\frac{\hbar^2}{2m^*R^2}(\frac{1}{i}\frac{\partial}{\partial\phi} + f + \Delta f)^2\Psi' = E\Psi'. \tag{4}$$

The wavefunctions after and before this process are related by

$$\Psi' = e^{-i\Delta f\phi}\Psi. \tag{5}$$

Boundary condition $\Psi'(\phi) = \Psi'(\phi + 2\pi n)$ implies that $\Psi(0) = e^{-i\Delta f 2\pi}\Psi(2\pi)$. Since the wavefunction must be single valued *not* all gauge transformations are allowed. Only those with integer Δf are possible. It should be noted that gauge transformations that change the flux *continuously* from zero to Φ_0 do *not* exist. The only possible gauge transformations are those that add an integral multiple of Φ_0[17], i.e., only *large* gauge transformations are possible. Under a large gauge transformation the wavefunction transforms

$$\Psi' = e^{-i\Delta f\phi}\Psi = e^{i(n-\Delta f)\phi} \tag{6}$$

The angular momentum has decreased by Δf. The gauge invariance requires that the energies before and after are equal: $E_n(f) = E_{n-\Delta f}(f + \Delta f)$. This should be contrasted with the adiabatic addition of flux from zero to one unit of quantum flux, which changes the electron energy from $E_0(n+f)^2$ to $E_0(n+f+1)^2$ while the wavefunction remains unchanged.

III. MODEL HAMILTONIAN, SYMMETRY OPERATIONS AND DEGENERATE SOLUTIONS

Let us now consider semiconductor rings with spin orbit terms. The electron is under the influence of the vector potential due to a solenoid at the center of the ring. The total Hamiltonian is $H = H_0 + H_R + H_D + H_Z$. The first part is $H_0 = \frac{\hbar^2 \vec{\Pi}^2}{2m^*} + U(r) + V(z)$, where $U(r)$ is the radial potential energy and $V(z)$ is the Rashba confinement potential. If the radius of the solenoid is equal or larger than the radius of the ring the electron experiences a magnetic field and the Zeeman term should be included: $H_Z = \frac{1}{2} g_0 \mu_B \sigma_z B$, where g_0, μ_B, and B are, respectively, the effective g-factor, the Bohr magneton, and the magnetic field. $\Pi_{x,y}$ are the kinematic momentum operators with $\Pi_{x,y} = k_{x,y} + \frac{e}{\hbar c} A_{x,y}$ and $k_x = \frac{1}{i} \frac{d}{dx}$. The x and y components of the vector potential are $A_x = -A_\phi sin\phi$ and $A_y = A_\phi cos\phi$. The Rashba spin orbit term is

$$H_R = c_R \left(\sigma_x \Pi_y - \sigma_y \Pi_x \right). \tag{7}$$

The constant c_R *depends on the external electric field E* applied along the z-axis. The Dresselhaus spin orbit term is

$$H_D = c_D \left(\sigma_x \Pi_x \left(\Pi_y^2 - \Pi_z^2 \right) + \sigma_y \Pi_y \left(\Pi_z^2 - \Pi_x^2 \right) \right). \tag{8}$$

There is another term of the form $\sigma_z \langle \Pi_z \rangle \left(\Pi_x^2 - \Pi_y^2 \right)$ in the Dresselhaus spin orbit term but it vanishes since the expectation value of the first subband wavefunction $f(z)$ along z-axis $\langle \Pi_z \rangle = \langle f(z)|k_z|f(z) \rangle = 0$. The constant c_D represents breaking of inversion symmetry by the crystal in zinc blende structures. The confinement potentials along the z-axis and radial direction are assumed to be sufficiently strong that only the lowest energy subbands are relevant. The total electron wavefunction in a narrow semiconductor ring can be written as $\Phi(\vec{r}) = f(z)R(r)\Psi(\phi)v(\vec{r})$. The lowest subband wavefunction along the radial direction is $R(r)$. The conduction band Bloch wavefunction is $v(\vec{r})$. In the following when the electron wavefunction is written only the azimuthal part $\Psi(\phi)$ will be shown and other wavefunctions will be suppressed.

The azimuthal part of the wavefunction can be written as

$$\Psi = \sum_{m=-\infty}^{\infty} c_{m\uparrow} e^{im\phi} | \uparrow > + \sum_{n=-\infty}^{\infty} c_{n\downarrow} e^{in\phi} | \downarrow >, \tag{9}$$

or in spinor notation $\Psi = \begin{pmatrix} F_\uparrow(\phi) \\ F_\downarrow(\phi) \end{pmatrix}$, where $F_\uparrow(\phi) = \sum_{m=-\infty}^{\infty} c_{m\uparrow} e^{im\phi}$ and $F_\downarrow(\phi) = \sum_{n=-\infty}^{\infty} c_{n\downarrow} e^{in\phi}$. Under time reversal operation[19] $\Psi \to \Psi'$, i.e.,

$$\begin{pmatrix} F_\uparrow(\phi) \\ F_\downarrow(\phi) \end{pmatrix} \to \begin{pmatrix} -F_\downarrow^*(\phi) \\ F_\uparrow^*(\phi) \end{pmatrix}, \tag{10}$$

or

$$\Psi' = K\psi = -\sum_m e^{-im\phi} c_{m\downarrow}^* | \uparrow \rangle + \sum_n e^{-in\phi} c_{n\uparrow}^* | \downarrow \rangle. \tag{11}$$

We have suppressed the Bloch wavefunction of the conduction band in applying the time reversal operator since it is unaffected by the operator K. Our wavefunctions are all effective mass wavefunctions and only the conduction band Bloch wavefunction at $\vec{k} = 0$ is relevant. Note that time reversal symmetry is *absent* when a vector potential is present.

Suppose that a solenoid with an integer or half integer fluxes, $k\Phi_0$, threads the quantum ring. The vector potential of the solenoid is $\vec{A}$. Consider a large gauge transformation that flips the direction of the vector potential: $\vec{A} \to \vec{A} - 2\vec{A} = -\vec{A}$. This operation is equivalent to the subtraction of the flux $2k\Phi_0$. If the flux $2k\Phi_0$ is *subtracted* first and time reversal operation is applied next we find that the wavefunction transforms as follows:

$$\begin{pmatrix} F_\uparrow(\phi) \\ F_\downarrow(\phi) \end{pmatrix} \to e^{i2k\phi} \begin{pmatrix} F_\uparrow(\phi) \\ F_\downarrow(\phi) \end{pmatrix} \to e^{-i2k\phi} \begin{pmatrix} -F_\downarrow^*(\phi) \\ F_\uparrow^*(\phi) \end{pmatrix}. \tag{12}$$

Note that this procedure is equivalent to applying time reversal operation first and *adding* the flux $2k\Phi_0$ next:

$$\begin{pmatrix} F_\uparrow(\phi) \\ F_\downarrow(\phi) \end{pmatrix} \to \begin{pmatrix} -F_\downarrow^*(\phi) \\ F_\uparrow^*(\phi) \end{pmatrix} \to e^{-i2k\phi} \begin{pmatrix} -F_\downarrow^*(\phi) \\ F_\uparrow^*(\phi) \end{pmatrix}. \tag{13}$$

From the Hamiltonian it can be seen that under the gauge transformation, $\vec{A} \to -\vec{A}$, and the time reversal operation, $\vec{k} \to -\vec{k}$ and $\vec{S} \to -\vec{S}$, the electron Hamiltonian is invariant in the absence of the Zeeman term. The angular momentum components of the wavefunctions before and after these operations are related as follows:

$$\Psi = \begin{pmatrix} \sum_m a_m e^{im\phi} \\ \sum_n b_n e^{in\phi} \end{pmatrix} \tag{14}$$

and

$$\Psi' = \begin{pmatrix} \sum_m A_m e^{im\phi} \\ \sum_n B_n e^{in\phi} \end{pmatrix}, \tag{15}$$

where $A_m = -b_{-m-2k}^*$ and $B_n = a_{-n-2k}^*$. These wavefunctions are degenerate. Unlike the case of the ideal one-dimensional ring the total degeneracy is now only two. The result given in this section is valid in the absence of the Zeeman term. However, a disorder potential and non-circular ring will not break the presence of degeneracies becuase the relevant potential $V_D(\vec{r})$ of these systems is invariant under the gauge transformation, $\vec{A} \to -\vec{A}$, and the time reversal operation, $\vec{k} \to -\vec{k}$ and $\vec{S} \to -\vec{S}$.

IV. ANALYTIC SOLUTIONS WHEN THE RASHBA SPIN ORBIT IS PRESENT

A narrow semiconductor ring with the Rashba spin orbit term can be solved exactly even in the presence of

the Zeeman term. The radial wavefunction $R(r)$ is taken to be a Gaussian and the correct Hamiltonian[20] is

$$
\begin{pmatrix}
E_0(\frac{1}{i}\frac{\partial}{\partial\phi} + f)^2 + z & E_R e^{-i\phi}(f - 1/2 + \frac{1}{i}\frac{\partial}{\partial\phi}) \\
E_R e^{i\phi}(f + 1/2 + \frac{1}{i}\frac{\partial}{\partial\phi}) & E_0(\frac{1}{i}\frac{\partial}{\partial\phi} + f)^2 - z
\end{pmatrix}. \tag{16}
$$

Here $E_R = \frac{c_R}{R}$. Note that the Zeeman term can be written as $z = g\mu_0 B/2 = z_0 f$, where $z_0 = g\mu B_0/2$ and $B_0 = \Phi_0/\pi R^2$. The eigenstates are of the form[6, 14, 15]

$$
\Psi_m = \begin{pmatrix} a_m e^{im\phi} \\ b_{m+1} e^{i(m+1)\phi} \end{pmatrix}. \tag{17}
$$

The coefficients a_m and b_{m+1} satisfy the matrix equation

$$
\begin{pmatrix}
E_0 p^2 + z & E_R(p + 1/2) \\
E_R(p + 1/2) & E_0(p + 1)^2 - z
\end{pmatrix}
\begin{pmatrix} a_m \\ b_{m+1} \end{pmatrix} = E \begin{pmatrix} a_m \\ b_{m+1} \end{pmatrix} \tag{18}
$$

with two eigenvalues

$$
E_\pm = \frac{1}{2}E_0(1 + 2p + 2p^2) \pm \frac{1}{2}X(E_R, p, z), \tag{19}
$$

where $p = f + m$ and

$$
\begin{aligned}
& X(E_R, p, z) \\
& = [(E_0^2 + E_R^2)(1 + 2p)^2 + 4z^2 - 4E_0 z(1 + 2p)]^{1/2}.
\end{aligned} \tag{20}
$$

The corresponding eigenvectors are

$$
\Psi_{m,\pm} = \begin{pmatrix} a_m^\pm e^{im\phi} \\ b_{m+1}^\pm e^{i(m+1)\phi} \end{pmatrix}, \tag{21}
$$

where

$$
\begin{pmatrix} a_m^\pm \\ b_{m+1}^\pm \end{pmatrix} = \frac{1}{N_\pm} \begin{pmatrix} -E_0 - 2E_0 p + 2z \pm X \\ E_R|1 + 2p| \end{pmatrix} \tag{22}
$$

with the normalization factors

$$
N_\pm = [E_R^2(1 + 2p)^2 + (E_0(1 + 2p) - 2z \mp X)^2]^{1/2}. \tag{23}
$$

A. Absence of the Zeeman term

When the Zeeman term is zero the expression for the eigenvalues simplifies

$$
\begin{aligned}
E_\pm &= \frac{1}{2}E_0(1 + 2p + 2p^2) \\
&\pm \frac{1}{2}\sqrt{(E_0^2 + E_R^2)(1 + 2p)^2}.
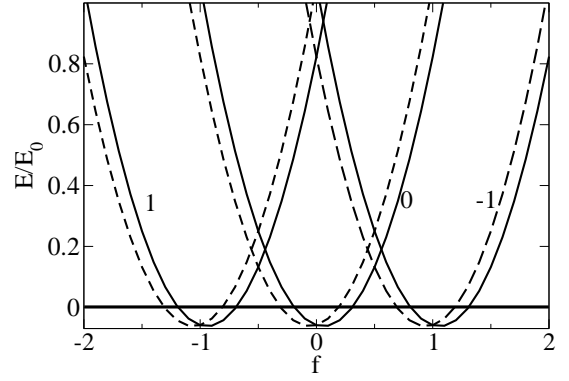\end{aligned} \tag{24}
$$

FIG. 2: Energy levels of a one-dimensional ring in the presence of the Rashba term $E_R = 0.5E_0$. Solid lines from right to left: $E_{-1,L}(f)$, $E_{0,L}(f)$, and $E_{1,L}(f)$. Dashed lines from right to left: $E_{-1,U}(f)$, $E_{0,U}(f)$, and $E_{1,U}(f)$.

These energies are not smooth functions of f. They can be combined into smooth functions $E_{m,L}(f)$ and $E_{m,U}(f)$ that are given by

$$
\begin{aligned}
E_{m,L}(f) &= \frac{1}{2}E_0(1 + 2p + 2p^2) - \frac{(1 + 2p)}{2}\sqrt{(E_0^2 + E_R^2)} \\
&= E_0(m + f - F_-)(m + f - F_+) \tag{25}
\end{aligned}
$$

and

$$
\begin{aligned}
E_{n,U}(f) &= \frac{1}{2}E_0(1 + 2k + 2k^2) + \frac{(1 + 2k)}{2}\sqrt{(E_0^2 + E_R^2)} \\
&= E_0(n + f - G_-)(n + f - G_+), \tag{26}
\end{aligned}
$$

where $k = n - 1 + f$. Note

$$
\begin{aligned}
F_\pm &= \frac{1}{2}(\sqrt{1 + (E_R/E_0)^2} - 1) \pm \frac{1}{2}E_R/E_0 \\
G_\pm &= -\frac{1}{2}(\sqrt{1 + (E_R/E_0)^2} - 1) \pm \frac{1}{2}E_R/E_0. \tag{27}
\end{aligned}
$$

The corresponding eigenvectors are

$$
\Psi_{m,L} = \begin{cases} \Psi_{m,+} & \text{for } 1 + 2(m + f) < 0 \\ \Psi_{m,-} & \text{for } 1 + 2(m + f) > 0 \end{cases} \tag{28}
$$

and

$$
\Psi_{n,U} = \begin{cases} \Psi_{n-1,-} & \text{for } 1 + 2(n - 1 + f) < 0 \\ \Psi_{n-1,+} & \text{for } 1 + 2(n - 1 + f) > 0. \end{cases} \tag{29}
$$

When $E_R = 0$ the constants $F_\pm = G_\pm = 0$ and $E_{m,L}(f) = E_{m,U}(f) = E_0(m + f)^2$. As a function of f the energies $E_{m,L}(f)$ and $E_{n,U}(f)$ describe parabola, as shown in Fig.(2). The roots of $E_{m,L}(f)$ are $f_- = F_- - m$ and $f_+ = F_+ - m$. The difference between them is E_R/E_0. The roots of $E_{n,U}(f)$ are $f_- = G_- - n$ and $f_+ = G_+ - n$. Again the difference between them is E_R/E_0.

Degeneracy at *half integer or integer* values of f occurs when the condition $E_{m,L}(f) = E_{n,U}(f)$ is satisfied. This leads to

$$
n = -m - 2f. \tag{30}
$$

At $f = 1/2$, for example, the pair represented by the quantum numbers $(m, n) = (0, -1)$ is degenerate: $E_{-1,U}(1/2) = E_{0,L}(1/2)$. The corresponding degenerate wavefunctions are

$$\Psi_{-1,U} = \Psi_{-2,-} = \begin{pmatrix} ae^{-2i\phi} \\ be^{-i\phi} \end{pmatrix} \qquad (31)$$

and

$$\Psi_{0,L} = \Psi_{0,-} = \begin{pmatrix} b \\ -ae^{i\phi} \end{pmatrix}, \qquad (32)$$

where a and b are real constants. They are related to each other by the subtraction of one unit of quantum flux and the application of time-reversal operation: $\Psi_{-1,U} = Ke^{i\phi}\Psi_{0,L}$. Note that in order to obtain this result we have multiplied the expression of $\Psi_{0,L}$, given by Eq.(21), with a phase factor.

B. Presence of Zeeman effect

Suppose the electron experiences an external magnetic field along the z-axis with the vector potential $\vec{A} = \frac{B}{2}(-y, x)$. To illustrate the effect of the Zeeman term we calculate the energy spectrum for $In_xGa_{1-x}As$ with the physical parameters $g = -4$ and $m^* = 0.05m$. For this semiconductor the Rashba term is the dominant spin orbit coupling. We assume that the radius is much larger than the ring width. For the radius of the ring $R = 14nm$ the energy scale is $E_0 = 3.9meV$ and $B[T] = 6.70f$. The Zeeman term couples the degenerate states $\Psi_{0,U}$ and $\Psi_{-1,L}$ since $< \Psi_{0,U}|H_Z|\Psi_{-1,L} >=< \Psi_{-1,-}|H_Z|\Psi_{-1,-} >$ is non zero. But it does not couple $\Psi_{-1,U}$ and $\Psi_{0,L}$ since $< \Psi_{0,L}|H_Z|\Psi_{-1,U} >=< \Psi_{0,-}|H_Z|\Psi_{-2,-} >$ is zero. This is the reason why there is only one anticrossing near, for example, $f = 1/2$, as can be see in Fig.(3). The value of degenerate energy is $E_0/4$ in the absence of the Zeeman term. Note that the degeneracy between $\Psi_{-1,L}$ and $\Psi_{-1,U}$ at $f = 1$ is lifted unlike the degeneracy between $\Psi_{0,L}$ and $\Psi_{0,U}$ at $f = 0$.

V. DISCUSSIONS

We have argued that the electron Hamiltonian of narrow semiconductor rings with the Rashba and Dresselhaus spin orbit terms is invariant under time-reversal operation followed by a large gauge transformation, provided that the Zeeman effect is absent. We find that all the eigenstates are doubly degenerate when integer or half-integer quantum fluxes thread the quantum ring. The wavefunctions of a degenerate pair are related to each other by the symmetry operation. These qualitative results are valid both for II-VI and III-V semiconductor rings. A disorder potential and non-circular ring will not break the relevant symmetry and the presence of
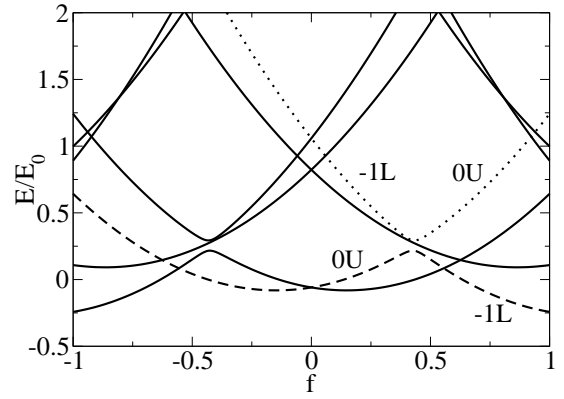

FIG. 3: Energy levels of a one-dimensional narrow ring in the presence of the Rashba interaction and the Zeeman term $E_R = 0.5E_0$. Anticrossing of the energy curves $E_{0,U}$ and $E_{-1,L}$ near $f = 0.5$. The labels on the dashed and dotted energy lines, $0U$ and $-1L$, indicate the correct quantum numbers in the absence of the Zeeman term.

degeneracy. We have obtained analytical and simple expressions for the energy spectrum when the Rashba term is dominant. Some of degenerate energy levels anticross in the presence of the Zeeman term.

Since the Zeeman effect complicates the energy spectrum to some degree, as shown in Fig.(3), semiconductor rings with a rather small Zeeman effect, such as GaAs rings, are more suitable for experimental study of the level degeneracy. However, in this case, a considerable numerical work is needed to compute a quantitatively correct energy spectrum due to the presence of the Dresselhaus term. Measurement of conductance oscillations[6, 11, 14, 15, 16] or optical emission lines[10] may reveal the level degeneracy.

Another suitable ring system to investigate is a superconductor and semiconductor hybrid system[21]. Each vortex of the Abrikosov lattice of the superconductor has half of the quantum unit flux, and a semiconductor ring under the superconductor can trap this flux[22]. The distance between the vortices must be larger than the diameter of the semiconductor ring, and this may be achieved by controlling the strength of the magnetic field. The advantage of such a system is that the Zeeman effect is absent since only the central part of the ring is threaded with half integer flux. When an InAs semiconductor is used, for example, the Rashba term is dominant, and our analytical results should describe well the energy spectrum of such a ring.

It is desirable to calculate numerically the energy spectrum quantitatively when both the Rashba and Dresselhaus terms are present in addition to the Zeeman term. It may be worthwhile to investigate the effect of the finite width of the ring [23]. The double degeneracy at an integer or half integer flux may be used to generate non-Abelian Berry phases, see Ref.19.

[1] S. Washburn and R.A. Webb, Adv. Phys. **35**, 395 (1986); A.G. Aronov and Y. V. Sharvin, Rev. Mod. Phys. **59**, 755 (1987).
[2] Y. Gefen, Y. Imry, and M. Y. Azbel, Phys. Rev. Lett. **52**, 129 (1984).
[3] M. Buttiker, Y.Imry, and R. Landauer, Phys.Lett, **96A**, 365 (1983).
[4] A. Lorke, R.J. Luyken, A. O. Govorov, J. Kotthaus, J.M. Garcia, and P.M. Petroff, Phys. Rev. Lett. **84**, 2223 (2000).
[5] Y.Meir, Y. Gefen, and O.Entin-Wohlman Phys. Rev. Lett. **63**, 798 (1989).
[6] A.G. Aronov and Y.B. Lyanda-Geller, Phys. Rev. Lett. **70**, 343 (1993).
[7] A.F. Morpurgo, J.P. Heida, T.M. Klapwijk, and B.J. van Wess, Phys. Rev. Lett. **80**, 1050 (1998).
[8] D.D. Awschalom, D. Loss, and N. Samarth, Semiconductor Spintronics and Quantum Computation (Springer, Berlin, 2002).
[9] R.J. Warburton, C. Schaflein, D. Haft, F. Bickel, A. Lorke, K. Karrai, J.M. Garcia, W. Schoenfeld, and P.M. Petroff, Nature **405**, 926 (2000).
[10] M. Bayer, M. Korkusinski, P. Hawrylak, T. Gutbrod, M. Michel, and Forchel, Phys. Rev. Lett. **90**, 186801 (2003).
[11] S. Souma and B. Nikolić, Phys. Rev. B, **70** 195346 (2004).
[12] J. Nitta, F.E. Meijer, and H. Takayanagi, Appl. Phys. Lett. **75**, 695 (1999).
[13] P. Földi, B. Molnár, M.G. Benedict, and F.M. Peeters, Phys. Rev. B, **71**, 033309 (2005).
[14] D. Frustaglia and K. Richter, Phys. Rev. B, **69** 235310 (2004).
[15] B. Molnár, F.M. Peeters, and P. Vasilopouos, Phys. Rev. B, **69** 155335 (2004).
[16] R. Capozza, D. Gilliano, P. Lucignano, and A. Tagliacozzo, Phys. Rev. Lett. **95**, 226803 (2005).
[17] D. Yoshioka, The Quantum Hall Effect (Springer, Berlin, 1998).
[18] E. Rashba cond-mat/0507007
[19] S.-R. Eric Yang and N.Y. Hwang, Phys. Rev. B, **73**, 125330 (2006).
[20] F.E. Meijer, A.F. Morpurgo, and T.M. Klapwijk, Phys. Rev. B, **66**, 033107 (2002).
[21] M. Berciu, T.G. Rappoport, and B. Janko, Nature **435**, 71 (2005).
[22] N. Byers and C.N. Yang, Phys. Rev. Lett. **7**, 46 (1961).
[23] C.W. Beenakker, H.van Houten, and A.A.M. Staring, Phys. Rev. B, **44**, 1657 (1991).